Suggested Solutions of HW 2

Ex 15.4

40. $V = 4\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{2\cos 2\theta}} r\sqrt{2-r^2}\,dr\,d\theta = -\frac{4}{3}\int_0^{\frac{\pi}{4}} \left[(2-2\cos 2\theta)^{\frac{3}{2}} - 2^{\frac{3}{2}}\right] d\theta$

$= \frac{2\pi\sqrt{2}}{3} - \frac{32}{3}\int_0^{\frac{\pi}{4}}(1-\cos^2\theta)\sin\theta\,d\theta = \frac{2\pi\sqrt{2}}{3} - \frac{32}{3}\left[\frac{\cos^3\theta}{3} - \cos\theta\right]_0^{\frac{\pi}{4}} = \frac{6\pi\sqrt{2}+40\sqrt{2}-64}{9}$

41. (a) $I^2 = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\frac{\pi}{2}}\int_0^\infty (e^{-r^2})\,r\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\left[\lim_{b\to\infty}\int_0^b re^{-r^2}dr\right]d\theta$

$= -\frac{1}{2}\int_0^{\frac{\pi}{2}}\lim_{b\to\infty}(e^{-b^2}-1)\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}}d\theta = \frac{\pi}{4}$. Thus $I = \frac{\sqrt{\pi}}{2}$

(b) $\lim_{x\to\infty}\int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}}\,dt = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}dt = \left(\frac{2}{\sqrt{\pi}}\right)\left(\frac{\sqrt{\pi}}{2}\right) = 1$, from part (a).

42. $\int_0^\infty\int_0^\infty \frac{1}{(1+x^2+y^2)^2}\,dx\,dy = \int_0^{\frac{\pi}{2}}\int_0^\infty \frac{r}{(1+r^2)^2}\,dr\,d\theta = \frac{\pi}{2}\lim_{b\to\infty}\int_0^b \frac{r}{(1+r^2)^2}\,dr = \frac{\pi}{4}\lim_{b\to\infty}\left[-\frac{1}{1+r^2}\right]_0^b$

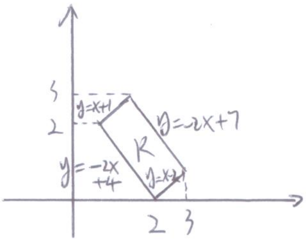
$= \frac{\pi}{4}\lim_{b\to\infty}\left(1-\frac{1}{1+b^2}\right) = \frac{\pi}{4}$

44. The area in polar coordinates is given by $A = \int_\alpha^\beta\int_0^{f(\theta)} r\,dr\,d\theta$

$= \int_\alpha^\beta\left[\frac{r^2}{2}\right]_0^{f(\theta)}d\theta = \frac{1}{2}\int_\alpha^\beta f^2(\theta)\,d\theta = \int_\alpha^\beta \frac{1}{2}r^2\,d\theta$, where $r = f(\theta)$.

Ex. 15.8

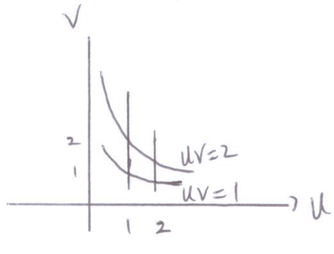
6. $\iint_R (2x^2-xy-y^2)\,dx\,dy = \iint_R (x-y)(2x+y)\,dx\,dy = \iint_G uv\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv = \frac{1}{3}\iint_G uv\,du\,dv$;



$\Rightarrow \frac{1}{3}\iint_G uv\,du\,dv = \frac{1}{3}\int_{-1}^2\int_4^7 uv\,dv\,du = \frac{1}{3}\int_{-1}^2 u\left[\frac{v^2}{2}\right]_4^7 du$

$= \frac{11}{2}\int_{-1}^2 u\,du = \frac{11}{2}\left[\frac{u^2}{2}\right]_{-1}^2 = \left(\frac{11}{4}\right)(4-1) = \frac{33}{4}$

10. (a) $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix}1 & 0\\ v & u\end{vmatrix} = u$,



(b) $x=1 \Rightarrow u=1$, and $x=2 \Rightarrow u=2$; $y=1 \Rightarrow uv=1 \Rightarrow v=\frac{1}{u}$, and $y=2$ $\Rightarrow uv=2 \Rightarrow v=\frac{2}{u}$. Then $\int_1^2\int_1^2 \frac{y}{x}\,dy\,dx = \int_1^2\int_{\frac{1}{u}}^{\frac{2}{u}}\left(\frac{uv}{u}\right)u\,dv\,du = \int_1^2\int_{\frac{1}{u}}^{\frac{2}{u}} uv\,dv\,du$

$= \int_1^2 u\left[\frac{v^2}{2}\right]_{\frac{1}{u}}^{\frac{2}{u}}du = \frac{3}{2}\left[\ln u\right]_1^2 = \frac{3}{2}\ln 2$.

$\int_1^2\int_1^2 \frac{y}{x}\,dy\,dx = \int_1^2\left[\frac{1}{x}\cdot\frac{y^2}{2}\right]_1^2 dx = \frac{3}{2}\int_1^2\frac{dx}{x} = \frac{3}{2}\ln 2$.

16. $x = u^2 - v^2$ and $y = 2uv$; $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4(u^2+v^2)$.

$y = 2\sqrt{1-x} \Rightarrow y^2 = 4(1-x) \Rightarrow (2uv)^2 = 4(1-(u^2-v^2)) \Rightarrow u = \pm 1$;

$y = 0 \Rightarrow 2uv = 0 \Rightarrow u = 0$ or $v = 0$; $x = 0 \Rightarrow u^2 - v^2 = 0 \Rightarrow u = v$ or $u = -v$.

$\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2+y^2}\, dx\, dy = \int_0^1 \int_0^u \sqrt{(u^2-v^2)^2+(2uv)^2} \cdot 4(u^2+v^2)\, dv\, du = 4\int_0^1 \int_0^u (u^2+v^2)^2\, dv\, du$

$= 4\int_1^2 [u^4 v + \frac{2}{3}u^2 v^3 + \frac{1}{5}v^5]_0^u\, du = \frac{112}{15}\int_1^2 u^5\, du = \frac{56}{45}$.

Advanced Ex. (898-900)

14. $\int_0^1 f(x)\left(\int_0^x g(x-y) f(y)\, dy\right) dx = \int_0^1 \int_0^x g(x-y) f(x) f(y)\, dy\, dx$

$= \int_0^1 \int_y^1 g(x-y) f(x) f(y)\, dx\, dy = \int_0^1 f(y)\left(\int_y^1 g(x-y) f(x)\, dx\right) dy$;

$\int_0^1 \int_0^1 g(|x-y|) f(x) f(y)\, dx\, dy = \int_0^1 \int_0^x g(x-y) f(x) f(y)\, dy\, dx$

$+ \int_0^1 \int_x^1 g(y-x) f(x) f(y)\, dy\, dx = \int_0^1 \int_y^1 g(x-y) f(x) f(y)\, dx\, dy + \int_0^1 \int_x^1 g(x-y) f(x) f(y)\, dy\, dx$

$= 2\int_0^1 \int_y^1 \cancel{g(y)} g(x-y) f(x) f(y)\, dx\, dy$

Thus $\int_0^1 f(x)\left(\int_0^x g(x-y) f(y)\, dy\right) dx = \frac{1}{2}\int_0^1 \int_0^1 g(|x-y|) f(x) f(y)\, dx\, dy$.

20. $\int_{y_0}^{y_1} \int_{x_0}^{x_1} \frac{\partial^2 F(x,y)}{\partial x\, \partial y}\, dx\, dy = \int_{y_0}^{y_1} \left[\frac{\partial F(x,y)}{\partial y}\right]_{x_0}^{x_1} dy = \int_{y_0}^{y_1} \left[\frac{\partial F(x_1,y)}{\partial y} - \frac{\partial F(x_0,y)}{\partial y}\right] dx$

$= [F(x_1,y) - F(x_0,y)]_{y_0}^{y_1} = F(x_1,y_1) - F(x_0,y_1) - F(x_1,y_0) + F(x_0,y_0)$.

22. (a) $\nabla f = x\mathbf{i} + y\mathbf{j} \Rightarrow D_u f = u_1 x + u_2 y$; and the region of area is $\frac{1}{2}$.

So average $= 2\int_0^1 \int_0^{1-x} (u_1 x + u_2 y)\, dy\, dx = 2\int_0^1 [u_1 x(1-x) + \frac{1}{2}u_2 (1-x)^2]\, dx$

$= 2(\frac{1}{6}u_1 + \frac{1}{6}u_2) = \frac{1}{3}(u_1+u_2)$.

(b) Average $= \frac{1}{\text{area}}\iint_R (u_1 x + u_2 y)\, dA = \frac{u_1}{\text{area}}\iint_R x\, dA + \frac{u_2}{\text{area}}\iint_R y\, dA$

$= u_1\left(\frac{M_y}{m}\right) + u_2\left(\frac{M_x}{m}\right) = u_1\bar{x} + u_2\bar{y}$.